# Lecture 20
# 15.1 Double and iterated integrals over rectangles

Jeremiah Southwick

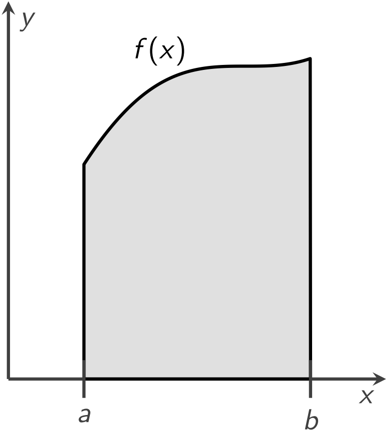
March 20, 2019

# Last class

Quadric Surfaces (12-6 lecture)

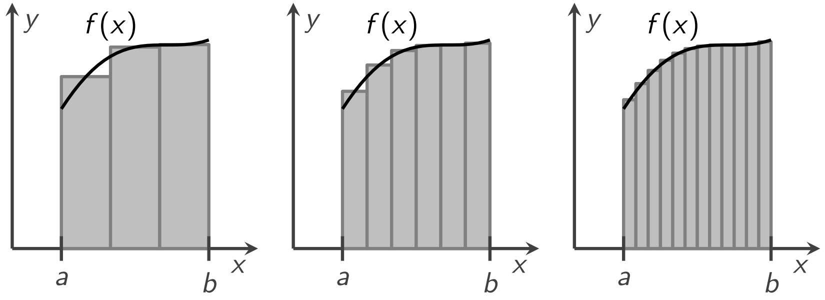
# Definite integrals

Recall that in Calculus we defined the integral from $x = a$ to $x = b$ of a function $f(x)$ to be the signed area under the function and above the $x$-axis.

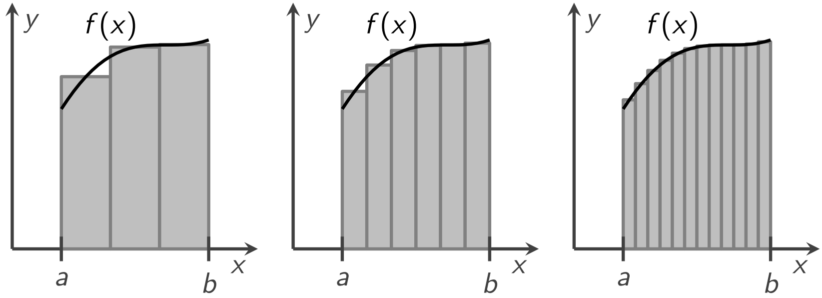


# Definite integrals

We approximated this area with rectangles of smaller and smaller width.



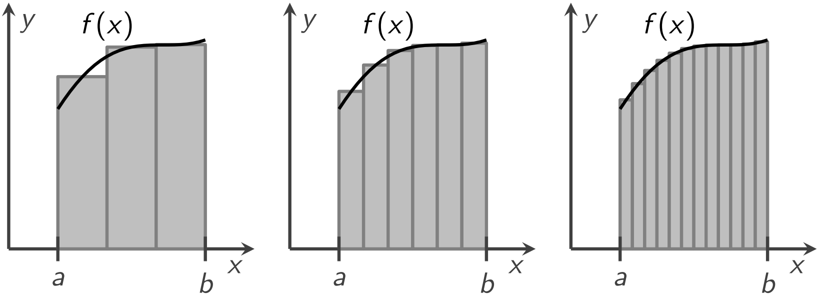
# Definite integrals




Then we defined the integral of the function from $x = a$ to $x = b$ to be the limit of the sum of the areas of the rectangles as the width of each rectangle went to 0:

$$\int_{x=a}^{x=b} f(x)dx =$$

# Definite integrals




Then we defined the integral of the function from $x = a$ to $x = b$ to be the limit of the sum of the areas of the rectangles as the width of each rectangle went to 0:

$$\int_{x=a}^{x=b} f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x_i$$

# Double integrals

We want to do the same thing with multivariable functions that we did with single variable functions in Calculus.

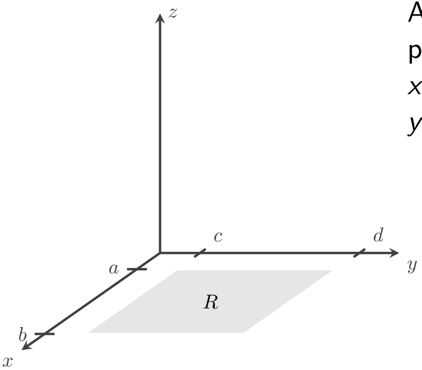
# Double integrals

We want to do the same thing with multivariable functions that we did with single variable functions in Calculus.
In Calculus we worked over an interval because our domain was the real line. Now we have to work over regions in the $xy$-plane. The most basic region is a rectangle. We'll call the rectangle $R$.

# Double integrals

We want to do the same thing with multivariable functions that we did with single variable functions in Calculus.

In Calculus we worked over an interval because our domain was the real line. Now we have to work over regions in the $xy$-plane. The most basic region is a rectangle. We'll call the rectangle $R$.

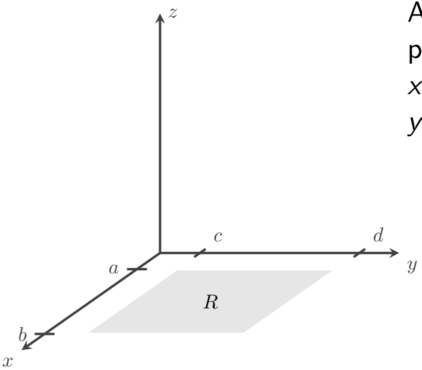



All rectangles are given by a pair of inequalities, one for the $x$-values and one for the $y$-values.

# Double integrals

We want to do the same thing with multivariable functions that we did with single variable functions in Calculus.

In Calculus we worked over an interval because our domain was the real line. Now we have to work over regions in the $xy$-plane. The most basic region is a rectangle. We'll call the rectangle $R$.




All rectangles are given by a pair of inequalities, one for the $x$-values and one for the $y$-values.

$$a \leq x \leq b \ \& \ c \leq y \leq d$$

# Double integrals

Given this framework, we can ask the following question:

## Question

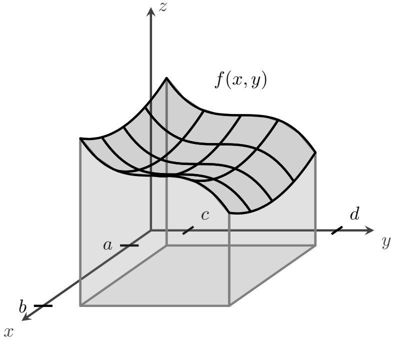
*How do we calculate the signed volume under a function and above a rectangle in the $xy$-plane?*

# Double integrals

Given this framework, we can ask the following question:

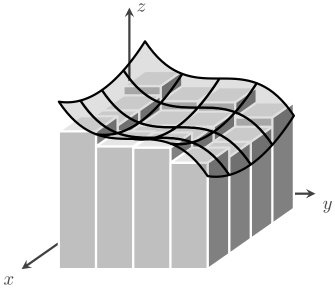
## Question

*How do we calculate the signed volume under a function and above a rectangle in the xy-plane?*



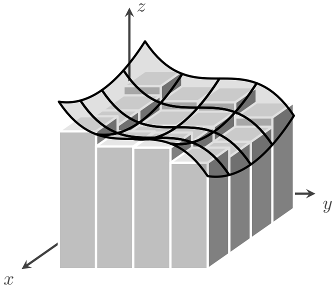
# Possible answer #1

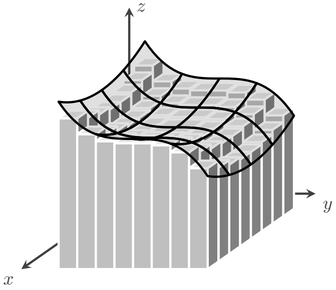
One way we could do this is to divide the rectangle up into smaller rectangles and approximate the volume from the resulting bricks.



# Possible answer #1

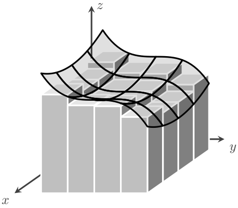
One way we could do this is to divide the rectangle up into smaller rectangles and approximate the volume from the resulting bricks.

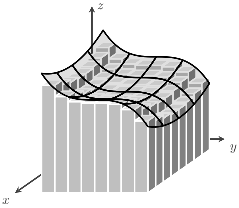



We could make these rectangles smaller and smaller and then the volume approximation would get better and better.

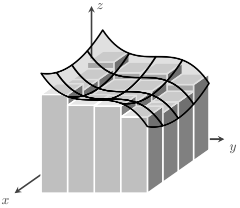
# Possible answer #1 cont.

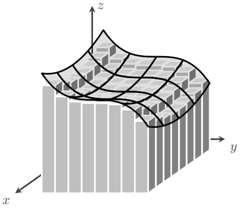



If we have $n$ smaller rectangles and a point $(x_i, y_i)$ in each rectangle, then the volume approximation would be

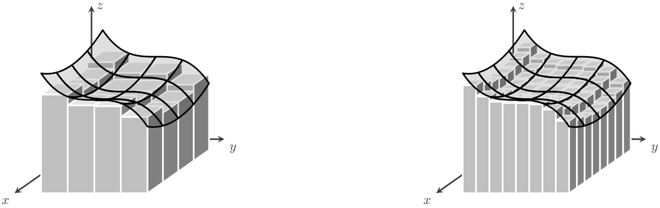
# Possible answer #1 cont.




If we have $n$ smaller rectangles and a point $(x_i, y_i)$ in each rectangle, then the volume approximation would be $\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$.

# Possible answer #1 cont.




If we have $n$ smaller rectangles and a point $(x_i, y_i)$ in each rectangle, then the volume approximation would be $\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$.
We define the double integral to be this limit as $n$ approaches infinity.

$$\text{Answer \#1: } \int\int_R f(x,y)dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

# Possible answer #2

In general, double integrals are hard to calculate. So instead we calculate volume as an *iterated integral*.

# Possible answer #2

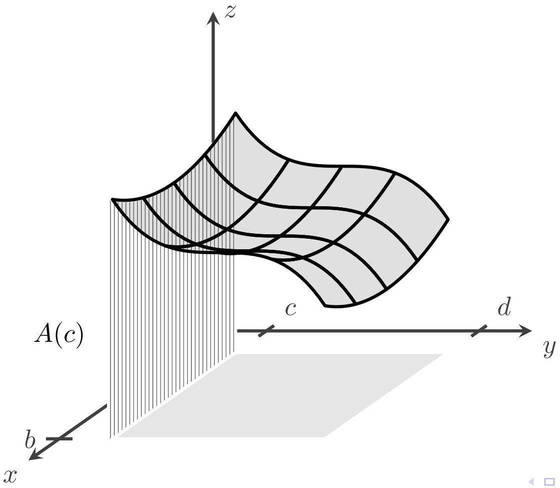
In general, double integrals are hard to calculate. So instead we calculate volume as an *iterated integral*.

The idea for an iterated integral is to first find a vertical slice of the volume. Let's take slices in the $x$-direction, starting at $y = c$.

# Possible answer #2

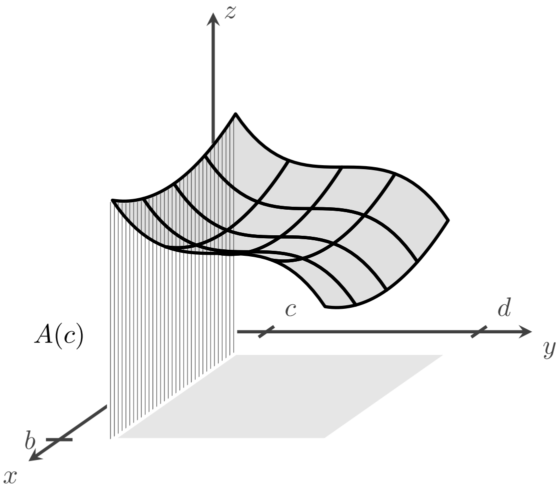
In general, double integrals are hard to calculate. So instead we calculate volume as an *iterated integral*.

The idea for an iterated integral is to first find a vertical slice of the volume. Let's take slices in the $x$-direction, starting at $y = c$.



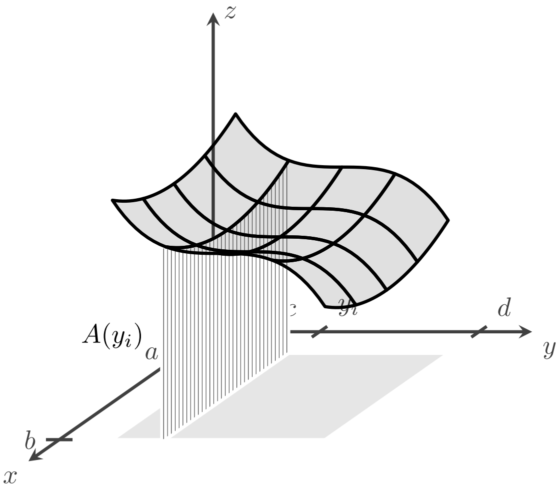
# Possible answer $\#2$ cont.

Next we will *iterate* the area we found across the region $R$ for different $y$-values. The area will be different for each $y$-value and hence will be a function of the $y$-value.



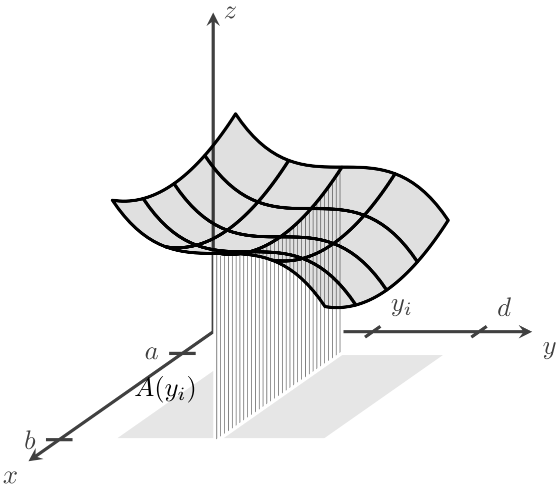
# Possible answer #2 cont.

Next we will *iterate* the area we found across the region $R$ for different $y$-values. The area will be different for each $y$-value and hence will be a function of the $y$-value.



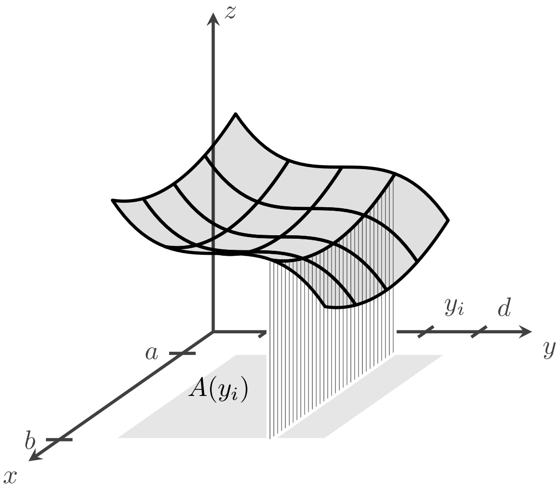
# Possible answer #2 cont.

Next we will *iterate* the area we found across the region $R$ for different $y$-values. The area will be different for each $y$-value and hence will be a function of the $y$-value.



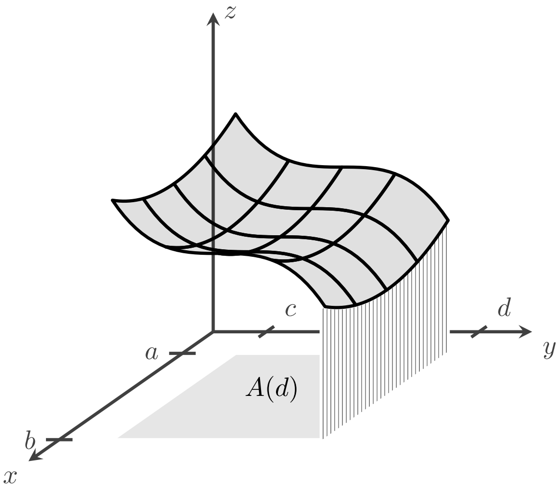
# Possible answer #2 cont.

Next we will *iterate* the area we found across the region $R$ for different $y$-values. The area will be different for each $y$-value and hence will be a function of the $y$-value.



# Possible answer #2 cont.

Next we will *iterate* the area we found across the region $R$ for different $y$-values. The area will be different for each $y$-value and hence will be a function of the $y$-value.



# Possible answer #2 cont.

If we continuously iterate this area-slice across the whole region from $y = c$ to $y = d$ and “add up” all the areas, we’ll get the volume of the region.

# Possible answer #2 cont.

If we continuously iterate this area-slice across the whole region from $y = c$ to $y = d$ and "add up" all the areas, we'll get the volume of the region.
Any time we "add up" a function like this, we are calculating an integral.

# Possible answer #2 cont.

If we continuously iterate this area-slice across the whole region from $y = c$ to $y = d$ and "add up" all the areas, we'll get the volume of the region.
Any time we "add up" a function like this, we are calculating an integral.

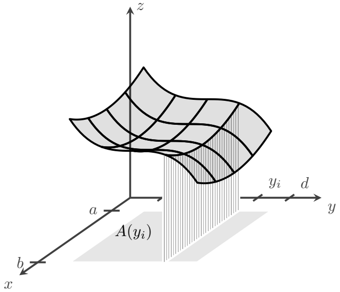
$$\text{Answer \#2: } \int_{y=c}^{y=d} A(y)dy$$

# Possible answer #2 cont.

$$\text{Answer \#2: } \int_{y=c}^{y=d} A(y)dy$$

## Question

*Can we find a formula for $A(y)$?*

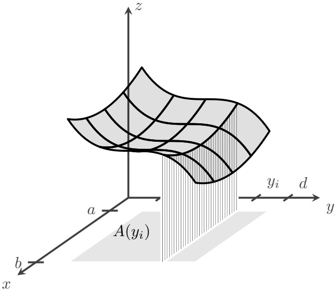


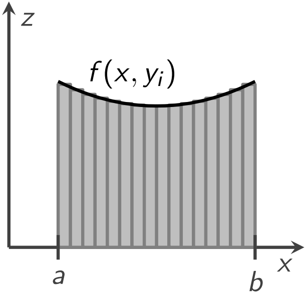
# Possible answer #2 cont.

$$\text{Answer \#2: } \int_{y=c}^{y=d} A(y)dy$$

### Question

*Can we find a formula for $A(y)$?*



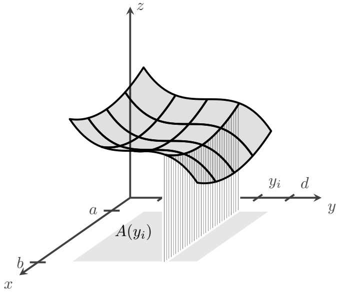



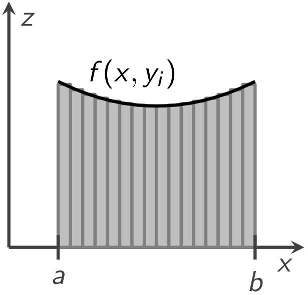
# Possible answer #2 cont.

$$\text{Answer \#2: } \int_{y=c}^{y=d} A(y)dy$$

## Question

*Can we find a formula for $A(y)$?*




So $A(y)$ is an integral. $A(y) = \int_{x=a}^{x=b} f(x,y)dx$.

# Answer #2 cont.

$$\text{Answer \#2: } \int_{y=c}^{y=d} A(y)dy \text{ and } A(y) = \int_{x=a}^{x=b} f(x,y)dx$$

This allows us to improve answer number 2.

# Answer #2 cont.

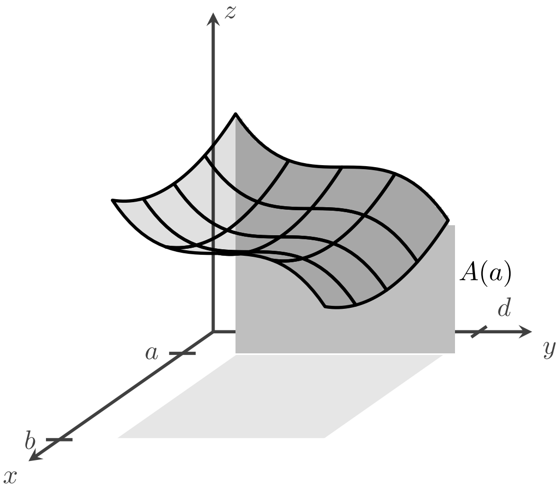
$$\text{Answer \#2: } \int_{y=c}^{y=d} A(y)dy \text{ and } A(y) = \int_{x=a}^{x=b} f(x,y)dx$$

This allows us to improve answer number 2.

$$\text{Answer \#2: } \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)dx\ dy$$

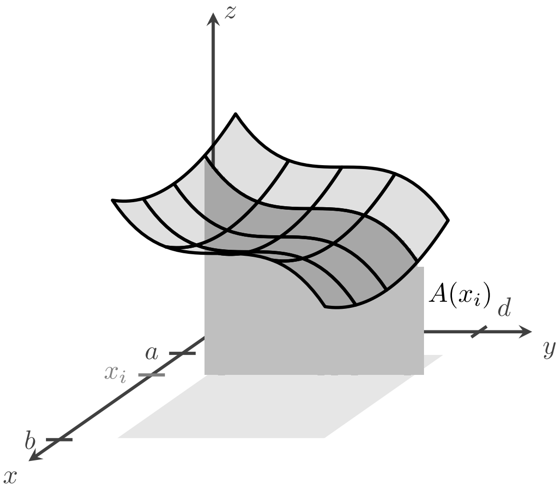
# Answer #3

There was nothing special about doing this in the $x$-direction first. If we had started with vertical slices in the $y$-direction, we could iterated through these:



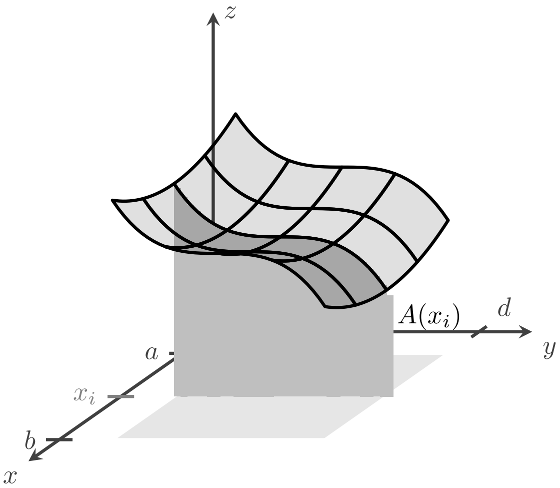
# Answer #3

There was nothing special about doing this in the $x$-direction first. If we had started with vertical slices in the $y$-direction, we could iterated through these:



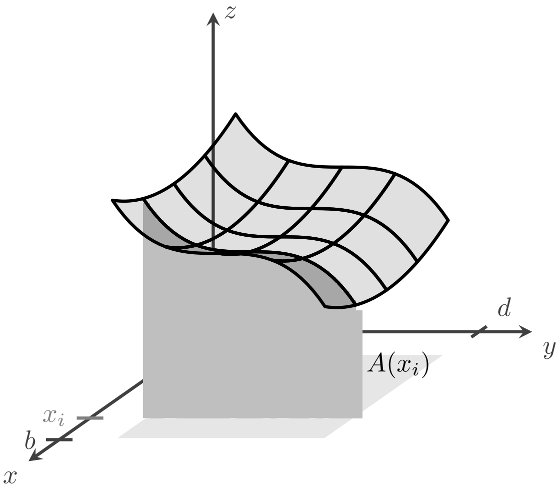
# Answer #3

There was nothing special about doing this in the $x$-direction first. If we had started with vertical slices in the $y$-direction, we could iterated through these:



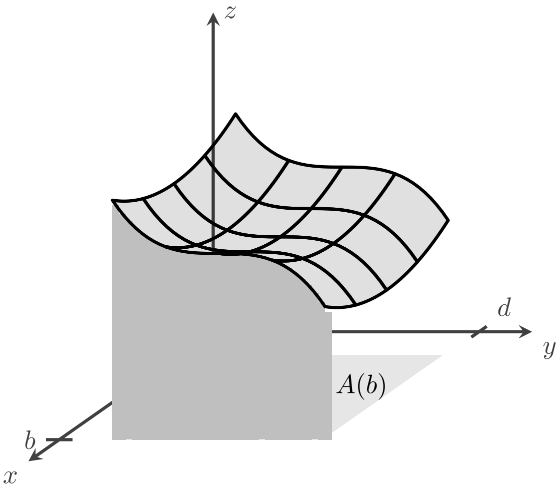
# Answer #3

There was nothing special about doing this in the *x*-direction first. If we had started with vertical slices in the *y*-direction, we could iterated through these:



# Answer #3

There was nothing special about doing this in the $x$-direction first. If we had started with vertical slices in the $y$-direction, we could iterated through these:



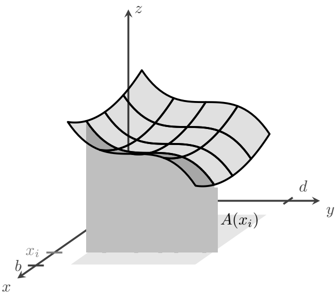
# Answer #3

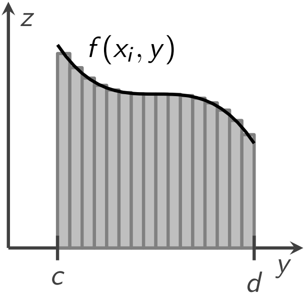
$$\text{Answer \#3: } \int_{x=a}^{x=b} A(x)dx$$

# Answer #3

$$\text{Answer \#3: } \int_{x=a}^{x=b} A(x)dx$$

As before, the area $A(x)$ is given by an integral.

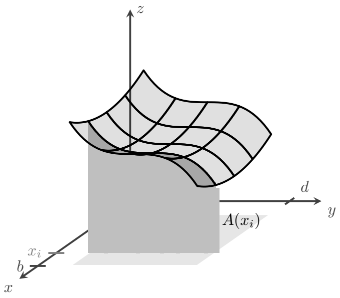


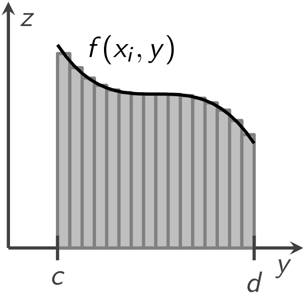



# Answer #3

$$\text{Answer \#3: } \int_{x=a}^{x=b} A(x)dx$$

As before, the area $A(x)$ is given by an integral.




$A(x) = \int_{y=c}^{y=d} f(x,y)dy.$

# Answer #3

Answer #2: $\int_{x=a}^{x=b} A(x)dx$ and $A(x) = \int_{y=c}^{y=d} f(x,y)dy$

Thus answer 3 looks like answer 2 but with the bounds switched.

# Answer #3

Answer #2: $\int_{x=a}^{x=b} A(x)dx$ and $A(x) = \int_{y=c}^{y=d} f(x,y)dy$

Thus answer 3 looks like answer 2 but with the bounds switched.

Answer #2: $\int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)dy\ dx$

# Fubini's Theorem

A mathematician named Fubini proved that all three of the answers we found are equal.

**Theorem**

*If $f(x, y)$ is continuous throughout the rectangular region $R : a \leq x \leq b, c \leq y \leq d$, then*

$$\int \int_R f(x, y) dA = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y) dx \ dy$$

$$= \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y) dy \ dx$$

# Example

## Example

*Find the volume under* $f(x, y) = 4 - x - y$ *and over the rectangle* $R : 0 \leq x \leq 2,\ 0 \leq y \leq 1$.

# Example

## Example

*Find the volume under* $f(x, y) = 4 - x - y$ *and over the rectangle* $R : 0 \leq x \leq 2,\ 0 \leq y \leq 1.$

We have volume =

$$\int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) dy\ dx = \int_{x=0}^{x=2} \left[ 4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx$$

$$= \int_{x=0}^{x=2} (4 - x - \frac{1}{2}) dx = \left[ \frac{7}{2}x - \frac{x^2}{2} \right]_{x=0}^{x=2} = \frac{7}{2}(2) - \frac{2^2}{2} = 5.$$

Or we could do it in the other order:

$$\int_{y=0}^{y=1} \int_{x=0}^{x=2} (4 - x - y) dx/dy = \int_{y=0}^{y=1} \left[ 4x - \frac{x^2}{2} - yx \right]_{x=0}^{x=1} dy$$

$$= \int_{y=0}^{y=1} [8 - 2 - 2y] dy = \left[ 6y - y^2 \right]_{y=0}^{y=1} = 6 - 1^2 = 5$$

# Another example

## Example

*Calculate* $\int\int_R xye^{xy^2}\, dA$ *for* $R : 0 \le x \le 2,\ 0 \le y \le 1.$

# Another example

## Example

*Calculate* $\int\int_R xye^{xy^2}\ dA$ *for* $R: 0 \leq x \leq 2,\ 0 \leq y \leq 1.$

If we choose $dxdy$, then we'll have to begin with a messy integration by parts calculation. However, if we begin $dydx$, then the exponent on the exponential has partial derivative $2xy$, which appears (without the 2) as a coefficient. Thus the order $dydx$ works as a $u - du$ substitution.

# Another example

## Example

*Calculate* $\int\int_R xye^{xy^2}\ dA$ *for* $R: 0 \le x \le 2,\ 0 \le y \le 1.$

If we choose $dxdy$, then we'll have to begin with a messy integration by parts calculation. However, if we begin $dydx$, then the exponent on the exponential has partial derivative $2xy$, which appears (without the 2) as a coefficient. Thus the order $dydx$ works as a $u - du$ substitution.

$$\int_{x=0}^{x=2}\int_{y=0}^{y=1} xye^{xy^2} dy\ dx = \frac{1}{2}\int_{x=0}^{x=2}\int_{y=0}^{y=1} 2xye^{xy^2} dy\ dx$$

$$= \frac{1}{2}\int_{x=0}^{x=2} e^{xy^2}\Big]_{y=0}^{y=1} dx = \frac{1}{2}\int_{x=0}^{x=2}(e^x - 1)dx$$

$$= \frac{1}{2}\left[e^x - x\right]_{x=0}^{x=2} = \frac{1}{2}[e^2 - 2 - 1 + 0] = \frac{e^2 - 3}{2}$$